

# ON OLIVER'S $p$ -GROUP CONJECTURE

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ABSTRACT. Let  $S$  be a  $p$ -group for an odd prime  $p$ . B. Oliver conjectures that a certain characteristic subgroup  $\mathfrak{X}(S)$  always contains the Thompson subgroup  $J(S)$ . We obtain a reformulation of the conjecture as a statement about modular representations of  $p$ -groups. Using this we verify Oliver's conjecture for groups where  $S/\mathfrak{X}(S)$  has nilpotence class at most two.

## 1. INTRODUCTION

The recently introduced concept of a  $p$ -local finite group seeks to provide a treatment of the  $p$ -local structure of a finite group  $G$  which does not refer directly to the group  $G$  itself and yet retains enough information to construct the  $p$ -localisation of the classifying space  $BG$ . Ideally one could then associate a  $p$ -local classifying space to a  $p$ -block of  $G$ , and to certain exotic fusion systems. See the survey article [1] by Broto, Levi and Oliver for an introduction to this area.

A key open question about  $p$ -local finite groups is whether or not there is a unique centric linking system associated to each saturated fusion system. Oliver showed that this would follow from a conjecture about higher limits (Conjecture 2.2 in [8]); and that for odd primes this higher limits conjecture would in turn follow from the following purely group-theoretic conjecture:

**Oliver's Conjecture 3.9.** ([8]) Let  $S$  be a  $p$ -group for an odd prime  $p$ . Then

$$J(S) \leq \mathfrak{X}(S),$$

where  $J(S)$  is the Thompson subgroup generated by all elementary abelian  $p$ -subgroups whose rank is the  $p$ -rank of  $S$ , and  $\mathfrak{X}(S)$  is the Oliver subgroup described in §2.

Our main result on Oliver's conjecture is as follows:

**Theorem 1.1.** *Let  $S$  be a  $p$ -group for an odd prime  $p$ . If  $S/\mathfrak{X}(S)$  has nilpotency class at most two, then  $S$  satisfies Oliver's conjecture.*

*Remark.* This subsumes all three cases of Oliver's Proposition 3.7 in the first case  $\mathfrak{X}(S) \geq J(S)$ .

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The proof of Theorem 1.1 depends on a reformulation of Oliver’s conjecture, for which we need to recall the terms  $F$ -module and offender. See e.g. [7] for a recent paper about offenders.

*Definition* (Definition 26.5 in [5]). Let  $G$  be a finite group and  $V$  a faithful  $\mathbb{F}_p G$ -module. If there exists a non-identity elementary abelian  $p$ -subgroup  $E \leq G$  which satisfies the inequality  $|E| |C_V(E)| \geq |V|$ , then  $V$  is called an  $F$ -module for  $G$ , and  $E$  an *offending subgroup*.

*Remark.*  $F$ -module is short for “failure of (Thompson) factorization module”. Another way to phrase the inequality is  $\dim(V) - \dim(V^E) \leq \text{rank}(E)$ .

We will always take  $G$  to be a nontrivial  $p$ -group. Hence the  $\mathbb{F}_p G$ -module  $V$  is faithful if and only if it is faithful as a module for  $\Omega_1(Z(G))$ . We shall be interested in the following stronger condition:

**(PS):** The restriction of  $V$  to each central order  $p$  subgroup has a nontrivial projective summand.

*Remark.* Projective and free are equivalent here. We are grateful to the referee for suggesting this formulation of the property. Another formulation is that every central order  $p$  element operates with minimal polynomial  $(X - 1)^p$ : equivalence follows from the standard properties of the Jordan normal form.

**Theorem 1.2.** *Let  $G \neq 1$  be a finite  $p$ -group. Then Oliver’s conjecture holds for every finite  $p$ -group  $S$  with  $S/\mathfrak{X}(S) \cong G$  if and only if  $G$  has no  $F$ -modules satisfying (PS).*

**Conjecture 1.3.** *Let  $p$  be an odd prime and  $G \neq 1$  a finite  $p$ -group. Then  $G$  has no  $F$ -modules which satisfy (PS).*

**Corollary 1.4.** *Conjecture 1.3 is equivalent to Oliver’s Conjecture 3.9.*

We prove Theorem 1.1 by verifying Conjecture 1.3 for groups of class at most two. For this we need the following result.

*Definition* (See [4]). Let  $V$  be a faithful  $\mathbb{F}_p G$ -module. A non-identity element  $g \in G$  is called *quadratic* if  $(g - 1)^2 V = 0$ .

**Theorem 1.5.** *Suppose that  $p$  is an odd prime,  $G$  is a  $p$ -group of nilpotence class at most two, and  $V$  is a faithful  $\mathbb{F}_p G$ -module. If  $G$  contains a quadratic element, then so does  $\Omega_1(Z(G))$ .*

*Structure of the paper.* We prove Theorem 1.2 and Corollary 1.4 in §2. In §3 we derive a consequence of the Replacement Theorem, Theorem 3.3. Then in §4 we prove Theorems 1.5 and 1.1. Finally in §5 we discuss a class three example which cannot be handled using Theorem 3.3.

## 2. THE REFORMULATION OF OLIVER'S CONJECTURE

For the convenience of the reader we start by recapping the definition and elementary properties of  $\mathfrak{X}(S)$ , as given in §3 of Oliver's paper [8].

*Definition* (c.f. [8], Def. 3.1). Let  $S$  be a  $p$ -group and  $K \triangleleft S$  a normal subgroup. A  $Q$ -series leading up to  $K$  consists of a series of subgroups

$$1 = Q_0 \leq Q_1 \leq \cdots \leq Q_n = K$$

such that each  $Q_i$  is normal in  $S$ , and such that

$$[\Omega_1(C_S(Q_{i-1})), Q_i; p-1] = 1$$

holds for each  $1 \leq i \leq n$ . The unique largest normal subgroup of  $S$  which admits such a  $Q$ -series is called  $\mathfrak{X}(S)$ , the Oliver subgroup of  $S$ .

**Lemma 2.1** (Oliver). *If  $1 = Q_0 \leq Q_1 \leq \cdots \leq Q_n = K$  is such a  $Q$  series and  $H \triangleleft G$  also admits a  $Q$ -series, then there is a  $Q$ -series leading up to  $HK$  which starts with  $Q_0, \dots, Q_n$ .*

*Hence there is indeed a unique largest subgroup admitting a  $Q$ -series, and this subgroup  $\mathfrak{X}(S)$  is characteristic in  $S$ . In addition,  $\mathfrak{X}(S)$  is centric in  $S$ : recall that  $P \leq S$  is centric if  $C_S(P) = Z(P)$ .*

*Proof.* See pages 334–5 of Oliver's paper [8]. □

Now we can start to derive the reformulation of Oliver's conjecture.

**Lemma 2.2.** *Let  $S$  be a finite  $p$ -group with  $\mathfrak{X}(S) < S$ . Then the induced action of  $G := S/\mathfrak{X}(S)$  on  $V := \Omega_1(Z(\mathfrak{X}(S)))$  satisfies (PS).*

*Proof.* Pick  $g \in S$  such that  $1 \neq g\mathfrak{X}(S) \in \Omega_1(Z(G))$ . Then  $\langle \mathfrak{X}(S), g \rangle \triangleleft S$  and so  $[V, g; p-1] \neq 1$ , by maximality of  $\mathfrak{X}(S)$ . So the minimal polynomial of the action of  $g$  does not divide  $(X-1)^{p-1}$ . But it has to divide  $(X-1)^p = X^p - 1$ . So  $(X-1)^p$  is the minimal polynomial. This is the reformulation of (PS). □

*Proof of Theorem 1.2.* Suppose first that no  $F$ -module for  $G$  satisfies (PS), and that  $S/\mathfrak{X}(S) \cong G$ . Let us prove Oliver's Conjecture for  $G$ . By Lemma 2.2 the induced action of  $G$  on  $V := \Omega_1(Z(\mathfrak{X}(S)))$  satisfies (PS), so by assumption there are no offending subgroups.

Let  $E \leq S$  be an elementary abelian subgroup not contained in  $\mathfrak{X}(S)$ . It suffices for us to show that  $\mathfrak{X}(S)$  contains an elementary abelian of greater rank than  $E$ . We can split  $E$  up as  $E = E_1 \times E_2 \times E_3$ , with  $E_1 = E \cap V \leq V^E$  and  $E_1 \times E_2 = E \cap \mathfrak{X}(S)$ . By assumption,  $1 \neq E_3$  embeds in  $S/\mathfrak{X}(S) \cong G$ . As there are no offenders, we have  $\dim(V) - \dim(V^{E_3}) > \text{rank}(E_3)$ . But  $V^{E_3} = V^E$ . So  $V \times E_2$  lies in  $\mathfrak{X}(S)$  and has greater rank than  $E$ .

Conversely suppose that the  $\mathbb{F}_p G$ -module  $V$  is an  $F$ -module and satisfies (PS). Set  $S$  to be the semidirect product  $S = V \rtimes G$  defined by this action. From Lemma 2.3 below we see that  $V = \mathfrak{X}(S)$ . As  $V$  is an  $F$ -module, there is an

offender: an elementary abelian subgroup  $1 \neq E \leq G$  with  $\dim(V) - \dim(V^E) \leq \text{rank}(E)$ . This means that  $W := V^E \times E$  is an elementary abelian subgroup which does not lie in  $V = \mathfrak{X}(S)$  but does have rank at least as great as that of  $\mathfrak{X}(S)$ . So  $W \leq J(S)$  and therefore  $J(S) \not\leq \mathfrak{X}(S)$ .  $\square$

**Lemma 2.3.** *Suppose that  $V$  is an  $\mathbb{F}_p G$ -module which satisfies (PS). Let  $S$  be the semidirect product  $S = V \rtimes G$  defined by this action. Then  $V = \mathfrak{X}(S)$ .*

*Proof.* First we prove that  $V$  is a maximal normal abelian subgroup of  $S$ : clearly it is abelian and normal. If  $A$  is a normal abelian subgroup strictly containing  $V$ , then  $A = V \rtimes H$  for some nontrivial abelian  $H \triangleleft G$ . As  $H$  is nontrivial and normal it contains an order  $p$  element  $g$  of  $Z(G)$ . Since  $V$  satisfies (PS), it follows that  $g$  acts on  $V$  with minimal polynomial  $(X - 1)^p$ . But that is a contradiction, as  $A$  is abelian. So  $V$  is indeed maximal normal abelian.

We now argue as in the proof of Oliver's Lemma 3.2. Since  $V$  is maximal normal abelian, it is centric in  $S$ : for if not then  $V < C_S(V) \triangleleft S$ , and so  $C_S(V)/V$  has nontrivial intersection with the centre of  $S/V$ . Picking an  $x \in C_S(V)$  whose image in  $C_S(V)/V$  is a nontrivial element of this intersection, we obtain a strictly larger normal abelian subgroup  $\langle V, x \rangle$ , a contradiction. Hence  $\Omega_1 C_S(V) = V$ .

Moreover, since  $V$  is normal abelian and  $p > 2$ , there is a  $Q$ -series  $1 < V$ . So by Lemma 2.1 there is a  $Q$ -series leading up to  $\mathfrak{X}(S)$  with  $Q_1 = V$ . If  $V < \mathfrak{X}(S)$  then there is  $Q_1 < Q_2 \triangleleft S$  with  $[V, Q_2; p - 1] = 1$ . But this cannot happen, because by the argument of the first paragraph of this proof there is a  $g \in Q_2$  whose action on  $V$  has minimal polynomial  $(X - 1)^p$ . So  $V = \mathfrak{X}(S)$ .  $\square$

*Proof of Corollary 1.4.* Immediate from Theorem 1.2. If  $\mathfrak{X}(S) = S$  then Oliver's Conjecture holds automatically.  $\square$

### 3. THE REPLACEMENT THEOREM

We shall need the following lemma, which is a special case of the Replacement Theorem and its proof in [6, X, 3.3].

**Lemma 3.1.** *Suppose that  $G \neq 1$  is elementary abelian, that  $V$  is a faithful  $\mathbb{F}_p G$ -module, and that  $G$  contains no quadratic elements. Let us write*

$$T = \{(H, W) \mid H \leq G \text{ and } W \text{ is a subspace of } V^H\}.$$

*Suppose that  $(H, W) \in T$  with  $H \neq 1$ . Then there is  $(K, U) \in T$  with  $K < H$ ,  $W \subsetneq U \subsetneq V$  and  $|H \times W| = |K \times U|$ .*

*Proof.* Let us set  $I = \{v \in V \mid (h - 1)v \in W \text{ for every } h \in H\}$  and  $J = \{v \in V \mid (h - 1)v \in I \text{ for every } h \in H\}$ . If  $1 \neq h \in H$  then  $(h - 1)^2 v \neq 0$  for some  $v \in V$ . Then  $v \notin I$ , for otherwise  $(h - 1)v \in W$  and so  $(h - 1)^2 v = 0$ . So  $I \subsetneq V$ , and therefore  $W \subsetneq I \subsetneq J$  by the usual orbit length argument. Pick  $v_0 \in J \setminus I$  and set  $U$  to be the subspace spanned by  $W$  and  $\{(h - 1)v_0 \mid h \in H\}$ . Set

$K = \{h \in H \mid (h-1)v_0 \in W\}$ . So  $U \supsetneq W$  by choice of  $v_0$ . Also  $U \subseteq I \subsetneq V$ . If  $h, h' \in H$  then  $(hh' - 1)v_0 = (h-1)v_0 + (h'-1)v_0 + (h-1)(h'-1)v_0$ , and so

$$(1) \quad (hh' - 1)v_0 \equiv (h-1)v_0 + (h'-1)v_0 \pmod{W}.$$

So  $K \leq H$ , and in fact  $K < H$  by choice of  $v_0$ . By Eqn. (1) it also follows that  $|H : K| = p^r$  for  $r = \dim U - \dim W$ . Finally  $U \subseteq V^K$ , for if  $k \in K$  and  $u \in U$ , then

$$u = \sum_{h \in H} \lambda_h (h-1)v_0 + w$$

for suitable  $\lambda_h \in \mathbb{F}_p$ ,  $w \in W$ . So

$$(k-1)u = \sum_{h \in H} \lambda_h (h-1)(k-1)v_0 = 0,$$

since  $(k-1)v_0 \in W \subseteq V^H$ . □

**Corollary 3.2.** *Suppose as in Lemma 3.1 that  $(H, W) \in T$  and  $H \neq 1$ . Then  $|H \times W| < |V|$ .*

*Proof.* By induction on  $|H|$ . By the lemma we may reduce  $|H|$  whilst keeping  $|H \times W|$  constant. This process only stops when we arrive at  $(K, U)$  with  $K = 1$ . But  $U \subsetneq V$  by the lemma. □

The following result is presumably well known to those familiar with Thompson factorization.

**Theorem 3.3.** *Suppose that  $p$  is an odd prime,  $G$  is a finite group,  $V$  is a faithful  $\mathbb{F}_p G$ -module, and  $E \leq G$  is a non-identity elementary abelian  $p$ -subgroup. If  $E$  is an offender, then it must contain a quadratic element.*

*Proof.* Without loss of generality  $E = G$ . Apply Corollary 3.2 to the pair  $(G, V^G) \in T$ . □

*Remark.* Pursuing this direction further, it might be worthwhile to investigate potential applications of the  $P(G, V)$ -theorem in the theory of  $p$ -local finite groups. The properties of the Thompson subgroup  $J(S)$  which Chermak describes in his comments on the motivation for the  $P(G, V)$ -theorem [2, Rk 2] are the same properties which led to  $J(S)$  featuring in Oliver's conjecture. And Timmesfeld's replacement theorem plays an important part in the proof of the  $P(G, V)$ -theorem.

#### 4. NILPOTENCE CLASS AT MOST TWO

We can now start work on the proof of Theorem 1.1.

**Lemma 4.1.** *Suppose that  $p$  is an odd prime, that  $G \neq 1$  is a finite  $p$ -group, and that  $V$  is a faithful  $\mathbb{F}_p G$ -module. Suppose that  $A, B \in G$  are such that  $C := [A, B]$  is a nontrivial element of  $C_G(A, B)$ . If  $C$  is non-quadratic, then so are  $A$  and  $B$ .*

*Proof.* By symmetry it suffices to prove that  $B$  is non-quadratic. So suppose that  $B$  is quadratic. Denote by  $\alpha, \beta, \gamma$  the action matrices on  $V$  of  $A - 1$ ,  $B - 1$  and  $C - 1$  respectively.

By assumption we have  $\gamma^2 \neq 0$  and  $\beta^2 = 0$ . As  $C$  commutes with  $A$  and  $B$ , we have  $\alpha\gamma = \gamma\alpha$  and  $\beta\gamma = \gamma\beta$ . Since  $[A, B] = C$ , we have  $AB = BAC$  and therefore

$$(2) \quad \alpha\beta - \beta\alpha = \gamma(1 + \beta + \alpha + \beta\alpha).$$

Evaluating  $\beta \cdot \text{Eqn. (2)} \cdot \beta$ , we deduce that  $\gamma\beta\alpha\beta = 0$ . So when we evaluate  $\beta \cdot \text{Eqn. (2)} + \text{Eqn. (2)} \cdot \beta$ , we find that  $\gamma(2\beta + \beta\alpha + \alpha\beta) = 0$ . Let us write  $\lambda = -\frac{1}{2}$  and  $\delta = \gamma\beta$ . Then we have

$$\delta = \lambda(\delta\alpha + \alpha\delta).$$

From this one sees by induction upon  $r \geq 1$  that

$$\delta = \lambda^r \sum_{s=0}^r \binom{r}{s} \alpha^s \delta \alpha^{r-s}.$$

As  $A$  has order a power of  $p$ , it follows that  $(A - 1)$  and its action matrix  $\alpha$  are nilpotent. From this we deduce that  $\delta = 0$ , that is  $\gamma\beta = 0$ . Applying this to  $\gamma \cdot \text{Eqn. (2)}$  we see that  $\gamma^2(1 + \alpha) = 0$ . As  $\alpha$  is nilpotent it follows that  $\gamma^2 = 0$ , a contradiction. So  $\beta^2 \neq 0$  after all.  $\square$

*Proof of Theorem 1.5.* We suppose that  $\Omega_1(Z(G))$  has no quadratic elements, and show that  $G$  has none either. Suppose  $1 \neq B \in Z(G)$ . Then is an  $r \geq 0$  with  $1 \neq B^{p^r} \in \Omega_1(Z(G))$ . So  $B^{p^r}$  is not quadratic. Hence  $(B - 1)^{2p^r} = (B^{p^r} - 1)^2$  has nonzero action. So  $(B - 1)^2$  has nonzero action, and  $Z(G)$  contains no quadratic elements.

If  $B \notin Z(G)$  then the nilpotency class is two and there is an element  $A \in G$  with  $1 \neq [A, B] \in Z(G)$ . So  $(B - 1)^2$  has nonzero action by Lemma 4.1.  $\square$

**Corollary 4.2.** *Suppose that  $p$  is an odd prime,  $G \neq 1$  a finite  $p$ -group and  $V$  an  $\mathbb{F}_p G$ -module which satisfies (PS). If the nilpotence class of  $G$  is at most two then  $V$  cannot be an  $F$ -module.*

*Proof.* As  $p$  is odd, condition (PS) means that there are no quadratic elements in  $\Omega_1(Z(G))$ . Then Theorem 1.5 says that there are no quadratic elements in  $G$ . So by Theorem 3.3 there are no offenders.  $\square$

*Proof of Theorem 1.1.* Follows from Corollary 4.2 and Theorem 1.2 if  $\mathfrak{X}(S) < S$ . If  $\mathfrak{X}(S) = S$  then there is nothing to prove.  $\square$

## 5. A CLASS 3 EXAMPLE

Theorem 1.5 was a key step in the proof of Theorem 1.1. We now give an example which shows that Theorem 1.5 does not apply to groups of nilpotence class three.

Let  $G$  be the semidirect product  $G = K \rtimes L$ , where the  $K = \mathbb{F}_3^3$  is elementary abelian of order  $3^3$ ,  $L = \langle A \rangle$  is cyclic of order 3, and the action of  $L$  on  $v \in K$  is given by

$$AvA^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \cdot v.$$

Observe that  $G$  is isomorphic to the wreath product  $C_3 \wr C_3$ , as the action of  $A$  permutes the following basis of  $K$  cyclically:  $(0, 0, 1)$ ,  $(0, 1, 1)$ ,  $(1, 2, 1)$ .

Setting  $B = (0, 0, 1)$ ,  $C = (0, 1, 0)$  and  $D = (1, 0, 0)$  we obtain the following presentation of  $G$ , where we take  $[A, B]$  to mean  $ABA^{-1}B^{-1}$ .

$$G = \left\langle A, B, C, D \mid \begin{array}{l} A^3 = B^3 = C^3 = D^3 = 1, \quad D \text{ central,} \\ [B, C] = 1, \quad [A, B] = C, \quad [A, C] = D \end{array} \right\rangle,$$

From this we deduce that matrices  $\alpha, \beta, \gamma, \delta \in M_n(\mathbb{F}_3)$  induce a representation  $\rho: G \rightarrow GL_n(\mathbb{F}_3)$  with

$$\rho(A) = 1 + \alpha \quad \rho(B) = 1 + \beta \quad \rho(C) = 1 + \gamma \quad \rho(D) = 1 + \delta$$

if and only if the following relations are satisfied, where  $[\alpha, \beta]$  now of course means  $\alpha\beta - \beta\alpha$ :

$$(3) \quad \begin{aligned} \alpha^3 &= \beta^3 = \gamma^3 = \delta^3 = 0 \\ [\alpha, \delta] &= [\beta, \delta] = [\gamma, \delta] = [\beta, \gamma] = 0 \\ [\alpha, \beta] &= \gamma(1 + \beta)(1 + \alpha) \quad [\alpha, \gamma] = \delta(1 + \gamma)(1 + \alpha) \end{aligned}$$

Now we consider what it means for such a representation to satisfy (PS). Here,  $Z(G) = \langle D \rangle$  is cyclic of order 3. So we need both  $(\rho(D) - 1)^2$  and  $(\rho(D^2) - 1)^2$  to be non-zero. That is,  $\delta^2$  and  $(\delta^2 + 2\delta)^2 = \delta^2(1 + \delta + \delta^2)$  should both be nonzero. But  $1 + \delta + \delta^2$  is invertible, since  $\delta$  is nilpotent.

We deduce therefore that matrices  $\alpha, \beta, \gamma, \delta \in GL_n(\mathbb{F}_3)$  induce a representation of  $G$  satisfying (PS) if and only if they satisfy the inequality

$$(4) \quad \delta^2 \neq 0$$

in addition to the equations (3).

Using GAP [3] we obtained the the following matrices in  $GL_8(\mathbb{F}_3)$ . The reader is invited to check<sup>1</sup> that they satisfy the relations (3) and (4).

$$\delta = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \gamma = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \alpha = \begin{pmatrix} 2 & 2 & 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Observe that  $\beta^2 = 0$ . So although this module satisfies (PS), the elementary abelian subgroups  $\langle B \rangle$  and  $\langle B, C, D \rangle$  both contain  $B$ , a quadratic element. So we must find another way to show that they are not offenders: Theorem 3.3 does not apply.

*Remark 5.1.* More generally, we are not currently able to decide Conjecture 1.3 either way for the wreath product group  $H \wr C_3$ , where the group  $H$  on the bottom is an elementary abelian 3-group.

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<sup>1</sup>See <http://www.minet.uni-jena.de/~green/Documents/matTest.g> for a GAP script which performs these checks.

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